Kriging

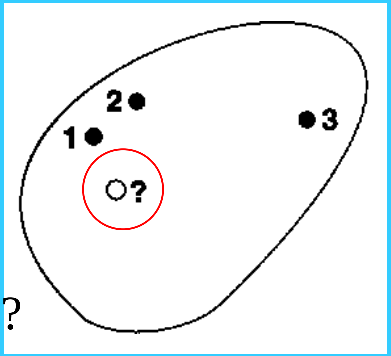
- Weighted Linear Estimators
- Some Definitions
- Derivation of the Kriging Equations
- Some Examples
- Different Types of Kriging
- How / Where is Kriging Used

Weighted Linear Estimators

- The basic idea is to estimate the attribute value (say, porosity) at a location where we do not know the true value

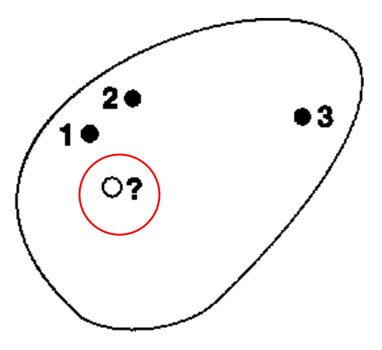
$$\mathbf{Z}^*(\mathbf{u}) = \sum_{\mathbf{i=1}}^{\mathbf{n}} \lambda_{\mathbf{i}} \cdot \mathbf{Z}(\mathbf{u}_{\mathbf{i}})$$

- where **u** refers to a location, Z*(**u**) is an estimate at location **u**, there are *n* data values $Z(\boldsymbol{u}_i)$, $i=1,\ldots,n$, and λ_i refer to weights.



- What factors could be considered in assigning the weights?
  - –closeness to the location being estimated
  - –redundancy between the data values
  - –anisotropic continuity (preferential direction)
  - –magnitude of continuity / variability

Weighted Linear Estimators



- Assign all of the weight to the nearest data (polygonal-type estimate)
- Assign the weights inversely proportional to the distance from the location being estimated (inverse distance schemes)

$$\lambda_i = \frac{\dfrac{1}{c + d_i^w}}{\sum_{i=1}^{n} \dfrac{1}{c + d_i^w}}$$

where d_i is the distance between data i and the location being estimated, c is a small constant, and ω is a power (usually between 1 to 3).

- How about using the variogram? $\mapsto$ that is kriging

Some Definitions

- Consider the residual data values:

  $$Y(\mathbf{u}_i) = Z(\mathbf{u}_i) - m(\mathbf{u}_i),\ i=1,\ldots,n$$

- where m(**u**) could be constant, locally varying, or considered constant but unknown.
- Variogram is defined as:

  $$2\,\gamma(\mathbf{h}) = \mathbf{E}\{[\ Y(\mathbf{u}\}) - Y(\mathbf{u} + \mathbf{h}]^2\}$$

- Covariance is defined as:

  $$C(\mathbf{h}) = \mathbf{E}\{\ Y(\mathbf{u}) \bullet Y(\mathbf{u} + \mathbf{h})\}$$

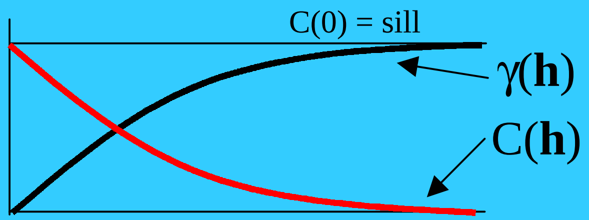
- Link between the Variogram and Covariance:

$$2\,\gamma(\mathbf{h}) = [\ E\{\ Y^2(\mathbf{u}) + [\ E\{\ Y^2(\mathbf{u} + \mathbf{h})\}] - 2 \bullet [\ E\{\ Y(\mathbf{u}) \bullet Y(\mathbf{u} + \mathbf{h})]$$

$$= Var\{Y(\mathbf{u})\} + Var\{Y(\mathbf{u} + \mathbf{h})\} - 2 \bullet C(\mathbf{h}\})$$

$$= 2\ [\ C(0) - C(\mathbf{h})]$$

So, $C(\mathbf{h}) = C(0) - \gamma(\mathbf{h})$



Simple Kriging (1)

- Consider a linear estimator:

$$\mathbf{Y^*(u) = \sum_{i=1}^{n} \lambda_i \cdot Y(u_i)}$$

- where $Y(\mathbf{u}_i)$ are the residual data (data values minus the mean) and $Y^*(\mathbf{u})$ is the estimate (add the mean back in)
- The error variance is defined as

A^2-2ab+b^2

$$E\{[Y^*(u) - Y(u)]^2\}$$

$$E\{[Y^*(u)]^2\} - 2 \cdot E\{Y^*(u) \cdot Y(u)\} + E\{[Y(u)]^2\}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j E\{Y(u_i) \cdot Y(u_j)\} - 2 \cdot \sum_{i=1}^{n} \lambda_i E\{Y(u) \cdot Y(u_i)\} + C(0)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j C(u_i, u_j) - 2 \cdot \sum_{i=1}^{n} \lambda_i C(u, u_i) + C(0)$$

Simple Kriging (2)

- Optimal weights $\lambda_i, i=1,\ldots,n$ may be determined by taking partial derivatives of the error variance w.r.t. the weights

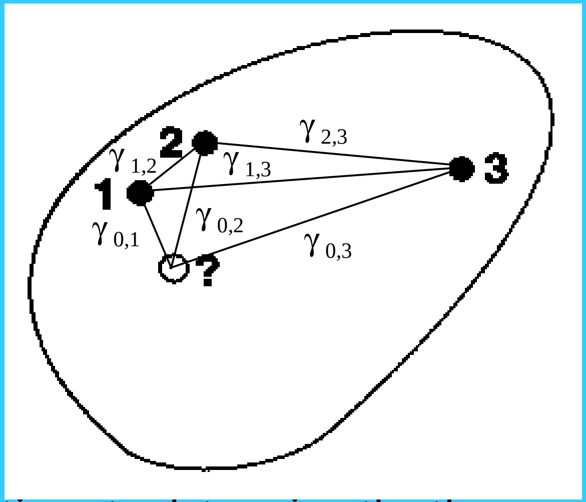
$$\frac{\partial[\quad]}{\partial\lambda_i} = 2\cdot\sum_{j=1}^{n}\lambda_j C(\mathbf{u}_i,\mathbf{u}_j) - 2\cdot C(\mathbf{u},\mathbf{u}_i), \quad i=1,\ldots,n$$

- and setting them to zero

$$\sum_{j=1}^{n}\lambda_j C(\mathbf{u}_i,\mathbf{u}_j) = C(\mathbf{u},\mathbf{u}_i), \quad i=1,\ldots,n$$

- This system of n equations with n unknown weights is the simple kriging (SK) system

Simple Kriging: Some Details



- There are three equations to determine the three weights:

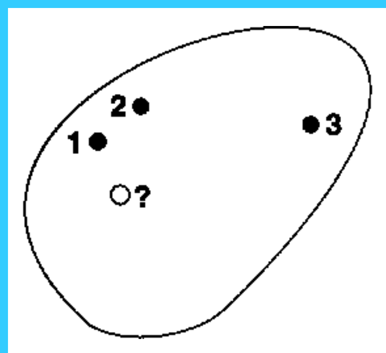
$$\lambda_1 \cdot C(1,1) + \lambda_2 \cdot C(1,2) + \lambda_3 \cdot C(1,3) = C(0,1)$$
$$\lambda_1 \cdot C(2,1) + \lambda_2 \cdot C(2,2) + \lambda_3 \cdot C(2,3) = C(0,2)$$
$$\lambda_1 \cdot C(3,1) + \lambda_2 \cdot C(3,2) + \lambda_3 \cdot C(3,3) = C(0,3)$$

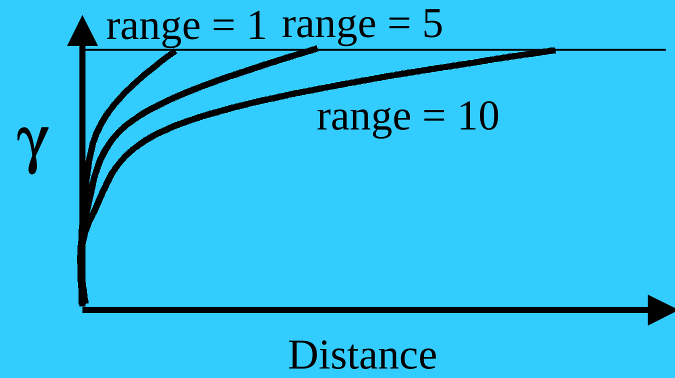
- In matrix notation: (Recall that C(**h**}) = C(0) - γ(**h**))

$$\begin{bmatrix} C(1,1) & C(1,2) & C(1,3) \\ C(2,1) & C(2,2) & C(2,3) \\ C(3,1) & C(3,2) & C(3,3) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} C(0,1) \\ C(0,2) \\ C(0,3) \end{bmatrix}$$

Simple Kriging

Changing the Range



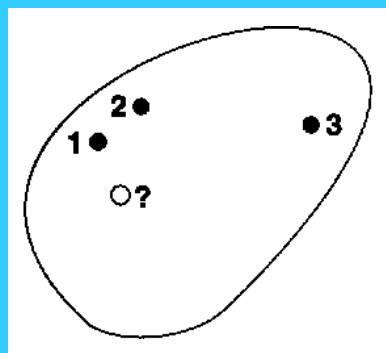


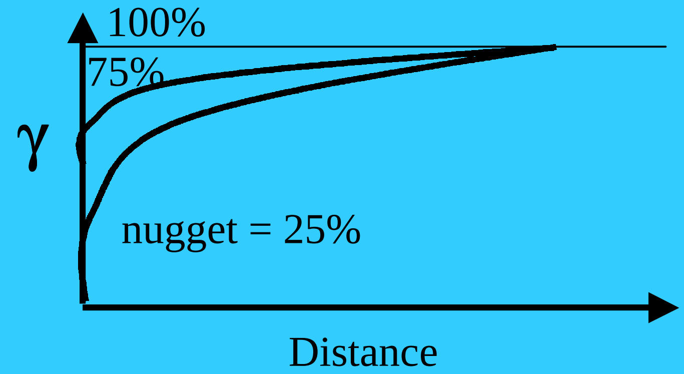
Simple kriging with a zero nugget effect and an isotropic spherical variogram with three different ranges:

	λ_1	λ_2	λ_3
range = 10	0.781	0.012	0.065
5	0.648	-0.027	0.001
1	0.000	0.000	0.000

Simple Kriging

Changing the Nugget Effect



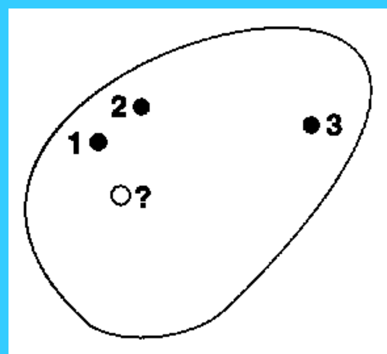


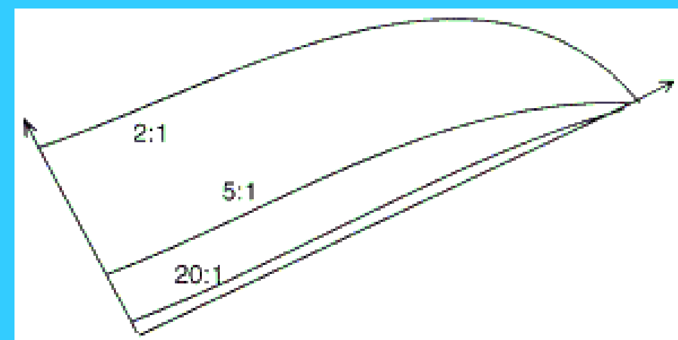
Simple kriging with an isotropic spherical variogram with a range of 10 distance units and three different nugget effects:

	λ_1	λ_2	λ_3
nugget = 0%	0.781	0.012	0.065
25%	0.468	0.203	0.064
75%	0.172	0.130	0.053
100%	0.000	0.000	0.000

Simple Kriging

Changing the Anisotropy





Simple kriging with a spherical variogram with a nugget of 25%, a principal range of 10 distance units and different “minor” ranges:

	λ_1	λ_2	λ_3
anisotropy 1:1	0.468	0.203	0.064
2:1	0.395	0.087	0.141
5:1	0.152	-0.055	0.232
20:1	0.000	0.000	0.239

Kriging

- All versions of kriging are elaborations on the basic linear regression algorithm and corresponding estimator:

$$[Z^*_{SK}(\mathbf{u}) - m(\mathbf{u})] = \sum_{\alpha=1}^{n} \lambda_\alpha(\mathbf{u})[Z(\mathbf{u}_\alpha) - m(\mathbf{u}_\alpha)]$$

- where Z(**u**) is the RV model at location **u**, the $\mathbf{u}_\alpha$'s are the *n* data locations, m(**u**) = E{Z(**u**) is the location-dependent expected value of RV Z(**u**), and Z_{SK}*(**u**) is the linear regression estimator, also called the “simple kriging” (SK) estimator.
- The SK weights λ_α (**u**) are given by the general non-stationary normal equations:

$$\sum_{\beta=1}^{n} \lambda_\beta(u) C(u_\beta, u_\alpha) = C(u, u_\alpha), \quad \alpha = 1, \ldots, n$$

- Some Remarks:
  - there are many types of kriging where specific constraints or methods of application are considered
  - the weights λ_α (**u**) account for (1) the proximity of the data to the location being estimated and (2) the clustering of the data
  - traditionally used for mapping
  - modern use is in the construction of local distributions of uncertainty for stochastic simulation algorithms

Kriging

- Minimum error variance estimate of the unknown:

$$z^*(\mathbf{u}) = \sum_{\alpha=1}^{n} \lambda_\alpha(\mathbf{u}) Z(\mathbf{u}_\alpha) + \left[1 - \sum_{\alpha=1}^{n} \lambda_\alpha(\mathbf{u})\right] m$$

- Simple kriging (SK): no constraint, the mean m is known.
- Ordinary kriging (OK): the sum of the weights $\sum_{\alpha=1}^{n} \lambda_\alpha(\mathbf{u})$ constrained to equal 1.
- Kriging with a trend model (KT) considers that m is unknown and that it has a more complex trend of known shape but unknown parameters.

$$m(\mathbf{u}) = \sum_{l=0}^{L} a_l f_l(\mathbf{u})$$

- where m($\mathbf{u}$) is the local mean, a_1, l = 0**...**,L are unknown coefficients of the trend model, and $f_l(\mathbf{u})$ are low order monomials of the coordinates. GSLIB allows nine different monomials up to quadratic functions of the coordinates (*x* , *y* , *z* , *xx* , *yy* , *zz*, *xy*, *xz* , *and*, *yz*).
- Kriging with an external drift is an extension of KT. Considers a single trend function $f_1(\mathbf{u})$ defined at each location from some external (secondary) variable.
- Cokriging is for kriging with different types of data.
- Indicator Kriging has a different goal $\mapsto$ to build a conditional cdf rather than derive an estimate.

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Parameter File For KT2D

```
                  Parameters for KB2D
                  *******************

 START OF PARAMETERS:
../data/cluster.dat              \ file with data
1   2   3                        \ columns for X, Y, and variable
-1.0e21   1.0e21                 \ trimming limits
3                                \ debugging level: 0,1,2,3
kb2d.dbg                         \ file for debugging output
kb2d.out                         \ file for kriged output
5    5.0  10.0                   \ nx,xmn,xsiz
5    5.0  10.0                   \ ny,ymn,ysiz
1    1                           \ x and y block discretization
4    8                           \ min and max data for kriging
20.0                             \ maximum search radius
1    2.302                       \ 0=SK, 1=OK,  (mean if SK)
1   2.0                          \ nst, nugget effect
1   8.0  0.0  10.0  10.0         \ it, c, azm, a_max, a_min
```

Parameter File For KT3D

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```
                     Parameters for KT3D
                     *******************

START OF PARAMETERS:
../data/cluster.dat          \ file with data
1   2   0    3    4          \ columns for X, Y, Z, var, sec var
-1.0e21   1.0e21             \ trimming limits
1                            \ option: 0=grid, 1=cross, 2=jackknife
xvk.dat                      \ file with jackknife data
1   2   0    3    0          \ columns for X,Y,Z,vr and sec var
3                            \ debugging level: 0,1,2,3
kt3d.dbg                     \ file for debugging output
kt3d.out                     \ file for kriged output
50   0.5    1.0              \ nx,xmn,xsiz
50   0.5    1.0              \ ny,ymn,ysiz
```

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Parameter File For KT3D

```
1    0.5    1.0                \ nz,zmn,zsiz
1    1      1                  \x,y and z block discretization
4    8                         \ min, max data for kriging
0                              \ max per octant (0-> not used)
20.0  20.0  20.0               \ maximum search radii
 0.0   0.0   0.0               \ angles for search ellipsoid
0     2.302                    \ 0=SK,1=OK,2=non-st SK,3=exdrift
0 0 0 0 0 0 0 0 0              \ drift: x,y,z,xx,yy,zz,xy,xz,zy
0                              \ 0, variable; 1, estimate trend
extdrift.dat                   \ gridded file with drift/mean
4                              \ column number in gridded file
1    0.2                       \ nst, nugget effect
1    0.8  0.0   0.0   0.0      \ it,cc,ang1,ang2,ang3
     10.0  10.0  10.0          \ a_hmax, a_hmin, a_vert
```

Kriging

- Kriging is a procedure for constructing a minimum error variance linear estimate at a location where the true value is unknown
- The main controls on the kriging weights are:
  - closeness of the data to the location being estimated
  - redundancy between the data
  - the variogram
- Simple Kriging (SK) does not constrain the weights and works with the residual from the mean
- Ordinary Kriging (OK) constrains the sum of the weights to be 1.0, therefore, the mean does not need to be known
- There are many different types of kriging
- Two implicit assumptions are stationarity (work around with different types of kriging) and ergodicity (more slippery)
- Kriging is not used directly for mapping the spatial distribution of an attribute (sometimes when the attribute is smooth). It is used, however, for building conditional distributions for stochastic simulation